

Is analytical dynamics a theoretical or an experimental science?

Firdaus E. Udwadia^{a,*}, Robert E. Kalaba^a, Yueyue Fan^b

^a*Department of Aerospace and Mechanical Engineering, School of Engineering, University of Southern California, 430 K Olin Hall, Los Angeles, CA 90089-1453, USA*

^b*School of Engineering, University of California, Davis, CA 95616, USA*

Abstract

When a mechanical system is subjected to equality constraints, use of the chain rule of differentiation and of generalized inverses of matrices enables us to write the most general possible equation of motion, no use being made of any physical principles, Eq. (8). Then employment of standard physical principles enables us to further interpret the terms in this general equation of motion.

© 2005 Elsevier Ltd. All rights reserved.

Keywords: Analytical dynamics; Principle of virtual work; Nonholonomic constraints; Nonideal constraints; Pseudoinverse of a matrix

1. Introduction

At first glance it might be assumed that the basic equation of motion of analytical dynamics would be replete with physical assumptions. Yet, this need not be so, as will unfold in the following pages.

Let us hurry to Eq. (8).

2. Statement of the problem

We consider a mechanical system consisting of p -point masses. The mass of the i th particle is denoted m_i , $i = 1, 2, \dots, p$. The position of this particle is the three-dimensional

* Corresponding author. Tel.: +1 213 740 0484; fax: +1 213 740 8071.

E-mail address: fudwadia@usc.edu (F.E. Udwadia).

column vector x_i , in an inertial Cartesian frame of reference. The system position vector is the column vector x ,

$$x = (x_1^T \ x_2^T \ \dots \ x_p^T)^T,$$

which is of dimension $3p = n$. We also introduce the mass matrix M , which is of dimension $3p \times 3p$, is a diagonal matrix, is positive definite, and has the masses $m_1, m_2, \dots, m_p$ down the main diagonal in groups of three, with zeros elsewhere. Assuming that the system is subjected to m equality constraints, involving t , the time, x , and x' , the system velocity vector, we wish first to determine *all* the possible equations of motion that are consistent with the constraints [9,10]. Following this, we shall introduce some physical assumptions and specialize the general explicit equation of motion to the physically relevant one.

3. Derivation of the basis equations

Suppose that the system is subjected to m nonholonomic or holonomic equality constraints of the form

$$f_i(x, x', t) = 0, \quad i = 1, 2, \dots, m. \quad (0)$$

Then by use of the chain rule of differentiation we arrive at a set of m equations that are *linear* in x'' , of the form,

$$Ax'' = b, \quad (1)$$

where A is an $m \times n = 3p$ matrix function of x , x' , and t , and b is an $m \times 1$ column vector that may depend upon x , x' , and t . Experience has shown that the matrix $AM^{-1/2}$ is of great significance [3,6,7], so we shall rewrite Eq. (1) as

$$AM^{-1/2}(M^{1/2}x'') = b. \quad (2)$$

The general solution of this equation may be written in the form [3,7]

$$M^{1/2}x'' = (AM^{-1/2})^+b + [I - (AM^{-1/2})^+(AM^{-1/2})]z, \quad (3)$$

where z is an arbitrary column vector of dimension $n = 3p$. The matrix $(AM^{-1/2})^+$ is the usual pseudoinverse of the matrix $AM^{-1/2}$ [1]. The first term on the right-hand side in Eq. (3), is a particular solution of Eq. (2), and the second is the general solution of the homogeneous equation $(AM^{-1/2})(M^{1/2}x'') = 0$.

We now choose to write the arbitrary vector z in the form

$$z = M^{1/2}a + M^{-1/2}c, \quad (4)$$

where a is a special vector that will be specified later, and c is an arbitrary vector. Eq. (3) then takes the form

$$M^{1/2}x'' = (AM^{-1/2})^+b + [I - (AM^{-1/2})^+AM^{-1/2}][M^{1/2}a + M^{-1/2}c], \quad (5)$$

which is

$$M^{1/2}x'' = (AM^{-1/2})^+b + M^{1/2}a + M^{-1/2}c - (AM^{-1/2})^+Aa - (AM^{-1/2})^+(AM^{-1/2})M^{-1/2}c. \quad (6)$$

Multiplying by $M^{-1/2}$, we find the desired equation of motion

$$x'' = a + M^{-1/2}(AM^{-1/2})^+(b - Aa) + M^{-1/2}[I - (AM^{-1/2})^+(AM^{-1/2})]M^{-1/2}c, \quad (7)$$

or alternatively [4,8],

$$Mx'' = Ma + M^{1/2}(AM^{-1/2})^+(b - Aa) + M^{1/2}[I - (AM^{-1/2})^+(AM^{-1/2})]M^{-1/2}c. \quad (8)$$

It is important to realize that the explicit equation of motion, either Eqs. (7) or (8), is the *most general possible* for motions that are consistent with the constraint equation, either Eqs. (0) or (1). No physical principle is involved in the derivation. The equation only involves mass, distance and time. We shall now investigate the three terms on the right-hand side of Eq. (8). We denote these by

$$F^N = Ma, \quad (9)$$

$$F^L = M^{1/2}(AM^{-1/2})^+(b - Aa) \quad (10)$$

and

$$F^C = M^{1/2}[I - (AM^{-1/2})^+AM^{-1/2}]M^{-1/2}c. \quad (11)$$

The notation recalls the names of Newton, Lagrange and Coulomb.

If there were no constraints, Eq. (7) shows that we would have $x'' = a$, so that the vector a is identified as the acceleration vector that the system would have if there were no constraints. If we now *introduce* the notion of “force”, F^N , then Newton’s second law is contained in Eq. (9), and we have $a = M^{-1}F^N$, where F^N is the impressed force vector.

To understand the force F^L in Eq. (10) we introduce the notion of “virtual displacement”, defined to be any vector v such that $Av = 0$. Then the “work” done by the force F^L in a virtual displacement v is $v^T F^L$, and we shall now see that this is zero. We have

$$AM^{-1/2}(M^{1/2}v) = 0. \quad (12)$$

From this it follows that

$$M^{1/2}v = [I - (AM^{-1/2})^+AM^{-1/2}]q, \quad (13)$$

where q is an arbitrary n -vector. Consequently,

$$v = M^{-1/2}[I - (AM^{-1/2})^+(AM^{-1/2})]q \quad (14)$$

and

$$v^T = q^T[I - (AM^{-1/2})^+(AM^{-1/2})]M^{-1/2}. \quad (15)$$

Thus, for the work done by the force F^L in any virtual displacement v , we have

$$v^T F^L = q^T [I - (AM^{-1/2})^+ (AM^{-1/2})] M^{-1/2} M^{1/2} (AM^{-1/2})^+ (b - Aa) = 0, \quad (16)$$

in view of the second condition for the generalized inverse $(AM^{-1/2})^+$,

$$(AM^{-1/2})^+ (AM^{-1/2}) (AM^{-1/2})^+ = (AM^{-1/2})^+. \quad (17)$$

This means that the constraint force F^L is the constraint force that maintains the constraints but does no work on the system in any virtual displacement [7]. It is the force that Lagrange himself safely characterized in the form $A^T \lambda$, where λ is an m -vector of unknown Lagrange multipliers [7]. Recall that A^T and A^+ have the same range spaces.

Finally, we consider the third term on the right-hand side of Eq. (8), F^C . How much work does F^C do in a virtual displacement v ? We have

$$\begin{aligned} v^T F^C &= q^T [I - (AM^{-1/2})^+ AM^{-1/2}] M^{-1/2} M^{1/2} [I - (AM^{-1/2})^+ AM^{-1/2}] M^{-1/2} c \\ &= q^T [I - (AM^{-1/2})^+ AM^{-1/2}] M^{-1/2} c \\ &= v^T c, \end{aligned} \quad (18)$$

since $I - (AM^{-1/2})^+ AM^{-1/2}$ is idempotent. From Eq. (18) we see that a specification of the arbitrary vector c is a specification of the work done by the constraint force F^C in a virtual displacement v . The total force of constraint is $F^L + F^C$, and since the force F^L does no work in a virtual displacement, we have

$$v^T (F^L + F^C) = v^T c. \quad (19)$$

Thus a specification of the vector c actually amounts to a specification of the amount of work to be done by the total constraint force $F^L + F^C$ in a virtual displacement v . Eq. (19) is a generalization of the classical principle of virtual work.

In classical analytical mechanics [11], following Lagrange, it is assumed that the constraint force does no work in a virtual displacement. This is referred to as an ideal constraint. This means that the fundamental assumption of classical analytical mechanics is that

$$F^C = 0, \quad (20)$$

so that the equation of motion, Eq. (8), reduces to [3,6,7]

$$Mx'' = Ma + M^{1/2} (AM^{-1/2})^+ (b - Aa). \quad (21)$$

More generally, though, as in situations in which sliding friction is significant, we shall have $F^C \neq 0$, in which case the more general equation of motion, Eq. (8), will apply. When $F^C \neq 0$, the constraint is referred to as nonideal.

On the mathematical side the derivation of the basic equation of motion, Eq. (8), involves the chain rule of differentiation and generalized inverses of matrices. On the physical side the notions of mass, distance and time occur. There is no mention of kinetic energy, potential energy, moments, etc. [2,5,11]. The concept of force occurs when we interpret the three terms on the right-hand side of Eq. (8). To further understand them, we introduce the concept of work. These are the only concepts introduced in this approach to analytical mechanics. In the applications to specific systems, of course, the customary centripetal and Coriolis

forces, moments, and so on do appear as in [7]. These motions emerge naturally from the terms in the right-hand side of Eq. (8), but no prior exposure to them is needed.

4. An example

A particle of unit mass is confined to the line $kx_1 - x_2 = 0$ in the vertical (x_1, x_2) plane. The acceleration of gravity in the negative x_2 direction is g . Suppose that the nonideal constraint engenders an electromagnetic force c that is given by the constant vector $(1 \ 2)^T$. Let us use Eq. (7) to obtain the equation of motion.

The mass matrix M is

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (22)$$

and the free acceleration vector a is

$$a = \begin{pmatrix} 0 \\ -g \end{pmatrix}. \quad (23)$$

From the constraint equation we see that

$$kx'_1 - x'_2 = 0, \quad (24)$$

and

$$kx''_1 - x''_2 = 0. \quad (25)$$

The last equation shows that A , a 1×2 matrix, is

$$A = (k \ -1) \quad (26)$$

and

$$b = 0, \quad (27)$$

a scalar. Thus

$$A^+ = (k^2 + 1)^{-1} \begin{pmatrix} k \\ -1 \end{pmatrix} \quad (28)$$

and

$$A^+ A = (k^2 + 1)^{-1} \begin{pmatrix} k^2 & -k \\ -k & 1 \end{pmatrix}, \quad (29)$$

$$I - A^+ A = (k^2 + 1)^{-1} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}. \quad (30)$$

Consequently, according to Eq. (7), the equation of motion is

$$x'' = \begin{pmatrix} 0 \\ -g \end{pmatrix} + I A^+ \left(0 - A \begin{pmatrix} 0 \\ -g \end{pmatrix} \right) + (k^2 + 1)^{-1} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad (31)$$

or

$$\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix} - \frac{g}{k^2 + 1} \begin{pmatrix} k \\ -1 \end{pmatrix} + \frac{1 + 2k}{k^2 + 1} \begin{pmatrix} 1 \\ k \end{pmatrix}. \quad (32)$$

The three terms on the right-hand side in the last equation are easily interpreted. The first is due to the force of gravity. The second maintains the constraint $x_2 = kx_1$ while doing no work on the particle in a virtual displacement. The third is due to the nonideal nature of the constraint. It is a constraint force that does the work $1 \, dx_1 + 2 \, dx_2$ in a virtual displacement for which $dx_2 = k \, dx_1$.

5. Discussion and comments

The explicit equation of motion, Eq. (8), is obtained using the same coordinates that the system is described in. There is no elimination of certain coordinates or introduction of generalized or quasi-coordinates. Also no Lagrange multipliers are introduced.

Since modern computing environments such as MATLAB or MATHEMATICA contain commands to obtain generalized inverses [7], Eq. (7) is most suitable for numerical implementation. A method for computing generalized inverses is given in [1].

Eq. (7) is obtained solely from the constraint conditions in Eqs. (0) or (1). Physical assumptions occur in determining the free acceleration vector ($a = M^{-1}F$). Virtual work considerations determine the vector c , most likely on the basis of experimental observations.

The derivation of Eq. (8) involves use of the chain rule of differentiation and the concept of the generalized inverse of a matrix. These suffice to determine the most general possible form of the equation of motion for systems that satisfy the m constraints

$$f_i(x, x', t) = 0, \quad i = 1, 2, \dots, m.$$

Notice that according to Eq. (11), if the vector $M^{-1/2}c$ lies in the null space of the matrix $AM^{-1/2}$, then $F^C = c$. On the other hand, if the vector $M^{-1/2}c$ lies in the column space of the matrix $(AM^{-1/2})^T$, then $F^C = 0$. Generally, of course, c will be composed of a sum of vectors in both spaces.

By the bold hypothesis that the constraint forces do no work in any virtual displacement, so that $F^C = 0$, Lagrange made it possible for analytical mechanics to be a theoretical rather than an experimental subject. He did away with the need to specify the vector c , which characterizes the work done in a virtual displacement v as $v^T c$. Only the constraints, as in Eqs. (0) or (1), are to be specified. There is no need to enquire further into the nonideal nature of the constraints and specify the work done by the constraint forces in a virtual displacement. For mechanical models to be more accurate, though, a more realistic specification of the vector c is mandatory. This will necessitate a return to experimental determination of the force vector c in many special cases, as the refractory nature of sliding friction dictates. Even after the vector c is specified, we must remember that not c but $M^{1/2}[I - (AM^{-1/2})^+ AM^{-1/2}]M^{-1/2}c$ is the term that actually enters the equation of motion, Eq. (8).

The right-hand side of Eq. (8) requires M , a , A , b , and c to be specified. The vector a is determined by Newton's law, $F^N = Ma$, or $a = M^{-1}F^N$. The matrix A and the vector b

are specified through the constraints, as in Eq. (1). Thus the specification of M , F^N , A and b completely determines the first two terms on the right-hand side of Eq. (8) ($a = M^{-1} F^N$). The specification of the third term is not complete until the vector c is specified (as a function of t , x , and x'). Since $v^T(F^L + F^C) = v^T c$, a specification of the vector c amounts to a specification of the amount of work done by the total force of constraint, $F^L + F^C$, in a virtual displacement v (for which $Av = 0$). Lagrange made the prescient assumption that in many instances we may set $F^C = 0$, so that the constraint forces do no work in a virtual displacement. Thus the knotty problem of dealing with c is eliminated. If more realism is required, then it becomes incumbent on the investigator to enquire more deeply into the actual form of c . Most likely this requires experimentation and the utilization of existing theories or the development of still others. One way or the other, the specification of an appropriate vector $c = c(t, x, x')$ is demanded.

References

- [1] Y. Fan, R.E. Kalaba, Dynamic programming and pseudo-inverses, *Appl. Math. Comput.* 139 (2003) 323–342.
- [2] H. Hertz, *The Principle of Mechanics*, Dover Publications, New York, 1956.
- [3] F.E. Udawadia, R.E. Kalaba, On motion, *J. Franklin Institute* 330 (1993) 571–577.
- [4] F.E. Udawadia, R.E. Kalaba, Non-ideal constraints and Lagrangian dynamics, *J. Aerospace Eng.* 13 (2000) 17–22.
- [5] C. Lanczos, *The Variational Principles of Mechanics*, University of Toronto Press, Toronto, 1970.
- [6] F.E. Udawadia, R.E. Kalaba, A new perspective on constrained motion, *Proc. Roy. Soc. London* 439 (1992) 407–410.
- [7] F.E. Udawadia, R.E. Kalaba, *Analytical Dynamics*, Cambridge University Press, New York, 1996.
- [8] F.E. Udawadia, R.E. Kalaba, Explicit equations of motion for mechanical systems with nonideal constraints, *J. Appl. Mech.* 68 (2001) 1–6.
- [9] F.E. Udawadia, R.E. Kalaba, On the foundations of analytical dynamics, *Int. J. Nonlinear Mech.* 37 (2002) 1079–1090.
- [10] F.E. Udawadia, R.E. Kalaba, What is the general form of the explicit equations of motion for constrained mechanical systems?, *J. Appl. Mech.* 69 (2002) 335–339.
- [11] E. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, New York, 1989 (original published in 1904).